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Dimension-eight operators in the weak OPE

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Abstract: We argue that there is a potential flaw in the standard treatment of weak decay amplitudes, including that of ϵ'/ϵ. We show that (contrary to conventional wisdom) dimension-eight operators *do* contribute to weak amplitudes, at order $G_F\alpha_s$ and without $1/M_W^2$ suppression. We demonstrate the existence of these operators through the use of a simple weak hamiltonian. Their contribution appears in different places depending on which scheme is adopted in performing the OPE. If one performs a complete separation of short and long distance physics within a cutoff scheme, dimension-eight operators occur in the weak hamiltonian at order $G_F\alpha_s/\mu^2$, μ being the separating scale. However, in an $\overline{\mathrm{MS}}$ renormalization scheme for the OPE the dimension-eight operators do not appear explicitly in the hamiltonian at order $G_F\alpha_s$. In this case, matrix elements must include physics above the scale μ, and it is here that dimension eight effects enter. The use of a cutoff scheme (especially quark model methods) for the calculation of the matrix elements of dimension-six operators is inconsistent with $\overline{\mathrm{MS}}$ unless there is careful matching including dimension-eight operators. The contribution of dimension-eight operators can be minimized by working at large enough values of the scale μ. We find from sum rule methods that the contribution of dimension-eight operators to the dimension-six operator $\mathcal{Q}_7^{(6)}$ is at the 100% level for $\mu = 1.5\,\mathrm{GeV}$. This suggests that presently available values of μ are too low to justify the neglect of these effects. Finally, we display the dimension-eight operators which appear within the Standard Model at one loop.

Keywords: Standard Model, Weak Decays, Kaon Physics.

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Contents

1. Introduction

The starting point for the study of nonleptonic weak transitions is the analysis of short-distance effects using perturbative QCD. The results are expressed using the Operator Product Expansion (OPE) as a series of local operators. In practice, only operators of dimension six are considered. We will argue that operators of dimension eight are also relevant, and that most previous analyses of nonleptonic amplitudes must be reconsidered.

Nonleptonic weak amplitudes represent probably the most difficult calculations in QCD. Since the W-boson propagator is a constant up to $Q \sim M_W$ in momentum space , the amplitude is sensitive to strong interaction physics at *all* energy scales. Therefore, one must control simultaneously the very low, intermediate and high energy portions of the calculation. There are two key ideas, both introduced by Wilson [1]–[4], that are used in this regard. One consists of separating the different energy scales and integrating out those effects from high energy. This yields an effective low energy theory with modified interactions. The second is the tool for doing this — the Operator Product Expansion (OPE) — in which the effects of high

(a) (b)

Figure 1: Scales in the weak transitions: (a) Long range, (b) Short range.

energy are replaced by local operators ordered according to increasing dimension. The latter can be applied at any scale, and as we reduce this scale we successively integrate out more and more physics, thereby changing the coefficients of the operators. Specifically, one can consider the physics above and below some energy scale μ. Throughout the paper, we will refer to μ as the *separation scale.* For example, if one takes the complete set of all local dimension-d operators[1] $\{\mathcal{Q}_i^{(d)}\}$ with the right quantum numbers, the operator product expansion tells us that a $\Delta S = 1$ amplitude can be written to leading order in dimension as

$$\langle \mathcal{H}_W^{(\Delta S=1)} \rangle = \frac{G_F}{\sqrt{2}}\, V_{us} V_{ud}^* \sum_d \sum_i \, \mathcal{C}_i^{(d)}(\mu)\, \langle \mathcal{Q}_i^{(d)} \rangle_\mu \, . \tag{1.1}$$

Here the $\{\mathcal{C}_i^{(d)}(\mu)\}$ are coefficients which describe the short distance physics with $Q \geq \mu$. The subscript 'μ' on the operator matrix element indicates that the matrix element is to include all physics up to the energy scale μ (i.e. with $Q \leq \mu$). The short distance OPE does not by itself solve the problem of nonleptonic amplitudes. However, it does tell us that the remaining task is to calculate the low energy matrix elements of local operators.

An example of this appears in figure 1, where the long-distance (low-energy) and short-distance (high-energy) parts of a nonleptonic weak transition are depicted separately. Let the separation scale be μ. Then the gluons shown in figure 1a have $Q \leq \mu$ and are associated with long-distance propagation. The blackened disc denotes all the (short-distance) effects with $Q \geq \mu$. We next look into the short-distance regime via figure 1b. Now there are only hard gluons with $Q \geq \mu$ which propagate over short-distances. If the separation scale μ is large enough, the physics at the higher energies can be analyzed with perturbative QCD. Because of this, the short-distance effects will appear local when viewed by low energy probes. Also, observe in figure 1b that the W-boson mass has been taken to infinity and so the process shown there corresponds to the range $\mu \leq Q \ll M_W$.

The most obvious candidates for the basis of weak nonleptonic operators are those with dimension six, formed as the product of two currents,

$$\bar{q}_1 \Gamma q_2 \bar{q}_3 \Gamma q_4 \, . \tag{1.2}$$

[1]Throughout this paper, we explicitly display the operator dimension as a superscript, e.g. $\mathcal{O}^{(d)}$.

Since the hamiltonian is the product of two weak currents, the coefficients of these operators are dimensionless. Next will come the operators of dimension eight, some examples of which are

$$\bar{q}_1 \Gamma \mathcal{D}_\mu q_2 \ \bar{q}_3 \Gamma \mathcal{D}^\mu q_4 \,, \tag{1.3}$$

where $\mathcal{D}_\mu q_i$ is the covariant derivative and

$$f_{abc} \bar{q}_1 \frac{\lambda^a}{2} \Gamma^\mu q_2 \ \bar{q}_3 \frac{\lambda^b}{2} \Gamma^\nu q_4 F^c_{\mu\nu} \,, \tag{1.4}$$

where $F^c_{\mu\nu}$ is the gluon field-strength tensor. The coefficients of these operators have engineering dimension $(\mathrm{Energy})^{-2}$. In the original papers [5, 6] on the weak interaction OPE it was stated explicitly that the coefficients of dimension-eight operators are of order $1/M_W^2$, and this has been accepted ever since. All the current treatments consider only operators of dimension six [7, 8, 9]. We will show that the correct procedure is more subtle.

Specifically, we will first consider the situation where one has a true separation of scales in the fashion outlined above. In this case:

1. Dimension-eight operators enter the weak OPE at order $1/\mu^2$ and not $1/M_W^2$.
2. In the one calculable example we know of, such effects continue to be significant even above the scale $\mu \sim 2\,\mathrm{GeV}$. Since most present calculations are performed with $\mu \sim 0.7 \to 2\,\mathrm{GeV}$, dimension-eight effects will likely affect the results of past work.
3. Most generally, the dimension-eight effect can appear in both the coefficient functions and the matrix elements. The relative amount of each depends on how one implements the division of physics at the scale μ and amounts to a 'separation scheme' dependence.

However, in the process of demonstrating these points, we will also see that dimensional regularization does not accomplish the separation of physics above and below the scale μ. Since dimensional regularization is by far the easiest calculational scheme, we study the structure of the OPE in such a scheme. We find:

4. Matrix element evaluations in dimensional regularization must be sensitive to energies *above* the scale μ.
5. In this case, the effects of dimension-eight operators appear fully within the matrix elements of dimension-six operators.
6. Mixed evaluations, in which one calculates the coefficients using dimensional regularization and the matrix elements using a form of a cutoff, are inherently inconsistent. Most past calculations fall in this category.

7. The influence of dimension-eight effects can be controlled by working at sufficiently large μ. Further work will be required to understand just how large μ must be to achieve a given precision.

This work has two basic parts. In the first, we use an explicit analytic calculation to illustrate the properties of dimension-eight operator effects in a weak amplitude. This will provide a demonstration of the above points. In the second part, we calculate the relevant dimension-eight operators for the Standard Model $\Delta S = 1$ weak hamiltonian in a particular separation scheme. This will allow the exploration of the size of such effects, provided the operator matrix elements can be evaluated on the lattice.

2. An explicit example

Rather than deal with the usual weak hamiltonian, we start with a similar but distinct operator that has simplified properties and allows us to demonstrate analytically the existence and properties of the dimension-eight operators. This hamiltonian [10, 11] contains one left-handed and one right-handed current instead of the usual Standard Model hamiltonian in which both currents are left-handed. Specifically we define[2]

$$\begin{aligned} \mathcal{H}_{\rm LR} &\equiv \frac{g_2^2}{8} \int d^4x \mathcal{D}_{\mu\nu}(x, M_W^2) J^{\mu\nu}(x)\,, \\ J^{\mu\nu}(x) &\equiv \frac{1}{2} T\left[\bar{d}(x)\Gamma_L^\mu u(x)\bar{u}(0)\Gamma_R^\nu s(0)\right] \\ &= \frac{1}{2} T\left[(V_{1-i2}^\mu(x) + A_{1-i2}^\mu(x))(V_{4+i5}^\nu(0) - A_{4+i5}^\nu(0))\right]\,, \end{aligned} \tag{2.1}$$

where $\mathcal{D}_{\mu\nu}$ is the W-boson propagator and V_a^μ, A_a^μ $(a = 1, \ldots, 8)$ are the flavor-octet vector, axialvector currents.

The reason why this hamiltonian provides a useful example is that in the chiral limit its matrix elements are related to vacuum matrix elements, and we may therefore take advantage of what is known about the associated vacuum polarization functions. [12, 13] For example, the K-to-pi matrix element

$$\mathcal{M}(p) = \langle \pi^-(p) | \mathcal{H}_{\rm LR} | K^-(p) \rangle \tag{2.2}$$

is given in the chiral limit of zero momentum and vanishing light-quark masses by the vacuum matrix element

$$\mathcal{M} \equiv \lim_{p=0} \mathcal{M}(p) = \frac{g_2^2}{16F_\pi^2} \int d^4x \mathcal{D}(x, M_W^2) \langle 0 | T\left(V_3^\mu(x) V_{\mu,3}(0) - A_3^\mu(x) A_{\mu,3}(0)\right) | 0 \rangle\,. \tag{2.3}$$

[2] We omit CKM dependence in the operator $\mathcal{H}_{\rm LR}$ and define the chiral matrices $\Gamma^\mu_{L \atop R} \equiv \gamma^\mu(1 \pm \gamma_5)$.

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Two of us have recently studied the amplitude $\mathcal{M}$ and more details on its properties can be found in ref. [10]. Here we will display those features useful for understanding the role of dimension-eight operators.

One can perform an operator product expansion on the hamiltonian $\mathcal{H}_{\rm LR}$ in the usual fashion. Including only the dimension-six operators one finds[3]

$$\mathcal{M} \simeq \frac{G_F}{2\sqrt{2}F_\pi^2}\left[c_1^{(6)}(\mu)\langle\mathcal{O}_1^{(6)}\rangle_\mu + c_8^{(6)}(\mu)\langle\mathcal{O}_8^{(6)}\rangle_\mu\right]. \tag{2.4}$$

The operator basis consists of two left-right operators $\mathcal{O}_1^{(6)}$, $\mathcal{O}_8^{(6)}$ which have respectively color-singlet and color-octet structure,

$$\begin{aligned}
\mathcal{O}_1^{(6)} &\equiv \bar{q}\gamma_\mu\frac{\tau_3}{2}q\ \bar{q}\gamma^\mu\frac{\tau_3}{2}q - \bar{q}\gamma_\mu\gamma_5\frac{\tau_3}{2}q\ \bar{q}\gamma^\mu\gamma_5\frac{\tau_3}{2}q\,,\\
\mathcal{O}_8^{(6)} &\equiv \bar{q}\gamma_\mu\lambda^a\frac{\tau_3}{2}q\ \bar{q}\gamma^\mu\lambda^a\frac{\tau_3}{2}q - \bar{q}\gamma_\mu\gamma_5\lambda^a\frac{\tau_3}{2}q\ \bar{q}\gamma^\mu\gamma_5\lambda^a\frac{\tau_3}{2}q\,.
\end{aligned} \tag{2.5}$$

In the above, $q = u, d, s$, τ_3 is a Pauli (flavor) matrix, $\{\lambda^a\}$ are the Gell Mann color matrices and the subscripts on $\mathcal{O}_1^{(6)}$, $\mathcal{O}_8^{(6)}$ refer to the color carried by their currents. The coefficient functions, including renormalization group summation, are

$$\begin{aligned}
c_1^{(6)}(\mu) &= \frac{1}{9}\left[\left(\frac{\alpha_s(\mu)}{\alpha_s(M_W)}\right)^{8/9} + 8\left(\frac{\alpha_s(\mu)}{\alpha_s(M_W)}\right)^{-1/9}\right],\\
c_8^{(6)}(\mu) &= \frac{1}{6}\left[\left(\frac{\alpha_s(\mu)}{\alpha_s(M_W)}\right)^{8/9} - \left(\frac{\alpha_s(\mu)}{\alpha_s(M_W)}\right)^{-1/9}\right],
\end{aligned} \tag{2.6}$$

with

$$\alpha_s(\mu) = \left[1 + 9\frac{\alpha_s(\mu)}{4\pi}\ln\left(\frac{M_W^2}{\mu^2}\right)\right]\alpha_s(M_W)\,. \tag{2.7}$$

For our purpose it is sufficient to work with an expansion of eq. (2.4) through first order in $\alpha_s(\mu)$,

$$\mathcal{M} \simeq \frac{G_F}{2\sqrt{2}F_\pi^2}\left[\langle\mathcal{O}_1^{(6)}\rangle_\mu + \frac{3}{8\pi}\ln\left(\frac{M_W^2}{\mu^2}\right)\langle\alpha_s\mathcal{O}_8^{(6)}\rangle_\mu + \cdots\right]. \tag{2.8}$$

Our goal is next to carry out an explicit evaluation of $\mathcal{M}$ and to demonstrate that dimension-eight operators appear in addition to those of dimension six.

2.1 The presence of dimension-eight operators

Let us analyze the vacuum matrix element that appears in eq. (2.3) in terms of the vacuum polarization function

$$\begin{aligned}
i\int d^4x e^{iq\cdot x}&\langle 0|T\left(V_3^\mu(x)V_3^\nu(0) - A_3^\mu(x)A_3^\nu(0)\right)|0\rangle =\\
&= (q^\mu q^\nu - q^2 g^{\mu\nu})(\Pi_{V,3} - \Pi_{A,3})(q^2) - q^\mu q^\nu \Pi_{A,3}^{(0)}(q^2)\,.
\end{aligned} \tag{2.9}$$

[3]We stress that $\{\mathcal{O}_k^{(6)}\}$ and $\{c_k^{(6)}\}$ of this section are distinct from $\{\mathcal{Q}_i^{(6)}\}$ and $\{\mathcal{C}_i^{(6)}\}$ of eq. (1.1).

Using this we transform the spatial integral in eq. (2.3) to momentum space,

$$\mathcal{M} = \frac{3G_F M_W^2}{32\sqrt{2}\pi^2 F_\pi^2} \int_0^\infty dQ^2 \; \frac{Q^4}{Q^2 + M_W^2} \left[\Pi_{V,3}(Q^2) - \Pi_{A,3}(Q^2)\right] . \tag{2.10}$$

Next we implement the separation of scales for the operator product expansion. We do this by applying a cutoff at $Q^2 = \mu^2$ and using the OPE in the high-energy/short-distance portion.

Thus consider a partition of $\mathcal{M}$ characterized by the scale μ,

$$\mathcal{M} = \mathcal{M}_<(\mu) + \mathcal{M}_>(\mu) , \tag{2.11}$$

where $\mathcal{M}_<(\mu)$ and $\mathcal{M}_>(\mu)$ are dependent respectively on contributions with $Q < \mu$ and $Q > \mu$. We obtain then for the low energy portion,

$$\begin{aligned} \mathcal{M}_<(\mu) &= \frac{3G_F M_W^2}{32\sqrt{2}\pi^2 F_\pi^2} \int_0^{\mu^2} dQ^2 \; \frac{Q^4}{Q^2 + M_W^2} \left[\Pi_{V,3}(Q^2) - \Pi_{A,3}(Q^2)\right] \\ &= \frac{3G_F}{32\sqrt{2}\pi^2 F_\pi^2} \int_0^{\mu^2} dQ^2 \, Q^4 \left[\Pi_{V,3}(Q^2) - \Pi_{A,3}(Q^2)\right] + \mathcal{O}\left(\frac{\mu^2}{M_W^2}\right) . \end{aligned} \tag{2.12}$$

This cut-off is well-defined as it refers to the external momentum of a gauge-invariant amplitude. We have shown in ref. [10] that this relation serves as a definition of the vacuum matrix element for the local operator $\mathcal{O}_1^{(6)}$ at the scale μ with a momentum-cutoff scheme (denoted as '(c.o.)'),

$$\langle \mathcal{O}_1^{(6)} \rangle_\mu^{(\mathrm{c.o.})} = \frac{3}{16\pi^2} \int_0^{\mu^2} dQ^2 \; Q^4 \left[\Pi_{V,3}(Q^2) - \Pi_{A,3}(Q^2)\right] . \tag{2.13}$$

More interesting for our purposes here is the high energy portion. We require that μ lie in the pQCD domain, and we further constrain it to obey $\mu \ll M_W$. The asymptotic behavior of the vacuum polarization operator is then described by the operator product expansion, involving a series of local operators ordered by increasing dimension. In the chiral limit the leading contribution to the difference of vector and axial-vector correlators is a four-quark operator of dimension six [12, 14], followed by a series of higher dimensional operators,

$$(\Pi_{V,3} - \Pi_{A,3})(Q^2) \sim \frac{2\pi \langle \alpha_s \mathcal{O}_8^{(6)} \rangle_\mu}{Q^6} + \frac{\mathcal{E}_\mu^{(8)}}{Q^8} + \cdots . \tag{2.14}$$

Here $\mathcal{E}_\mu^{(8)}$ represents the combination of local operators carrying dimension eight.

These have been discussed and partially calculated by Broadhurst and Generalis [15]. For our purposes, it is not necessary to know their specific form, but only the fact of their existence. Upon performing the integration over Q^2 at high energies, we find

$$\mathcal{M}_>(\mu) = \frac{3G_F}{32\sqrt{2}\pi^2 F_\pi^2} \left[\ln\left(\frac{M_W^2}{\mu^2}\right) 2\pi \langle \alpha_s \mathcal{O}_8^{(6)} \rangle_\mu + \frac{\mathcal{E}_\mu^{(8)}}{\mu^2} + \cdots \right] . \tag{2.15}$$

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In this expression we have dropped corrections of order μ^2/M_W^2.

The full amplitude is then

$$\mathcal{M} \simeq \frac{G_F}{2\sqrt{2}F_\pi^2}\left[\langle\mathcal{O}_1^{(6)}\rangle_\mu^{(\text{c.o.})} + \frac{3}{8\pi}\ln\left(\frac{M_W^2}{\mu^2}\right)\langle\alpha_s\mathcal{O}_8^{(6)}\rangle_\mu + \frac{3}{16\pi^2}\frac{\mathcal{E}_\mu^{(8)}}{\mu^2} + \cdots\right]. \qquad (2.16)$$

The crucial features here are the presence of the dimension-eight operators in the short distance portion of the amplitude $\mathcal{M}$ and the fact that they appear divided by the scale μ^2 instead of M_W^2. They are *not* suppressed by inverse powers of M_W because these operators appear in the vacuum polarization function at any Q^2 between μ^2 and ∞. From this calculation it is clear that these operators must be present in an OPE that describes the integrating-out of short distance physics. These operators have previously been missed in the usual treatment of the operator product expansion within the weak hamiltonian, although they have been properly included in the OPE for the vacuum polarization functions [12]. It will be clear from the work that we do below, where we find dimension-eight operators in the weak OPE, that they appear whenever we separate physics above and below the scale μ. Finally, despite our emphasis on the dimension-eight operators in this paper, operators of even higher dimension could play a role for sufficiently small values of μ. For example, the next term in eq. (2.16) would be $3\mathcal{E}_\mu^{(10)}/(32\pi^2\mu^4)$, where $\mathcal{E}_\mu^{(10)}$ represents the dimension-ten effect.

2.2 Estimated size

Here we give some numerical estimates of the size of the dimension-eight effects. This can be done using experimental data since the vacuum polarization functions satisfy dispersion relations. The inputs to the dispersion integrals, i.e. the imaginary parts are known from experimental work on cross section measurements of $e^+e^- \to$ hadrons and from the study of hadronic final states appearing in τ decay. We have performed the required phenomenology in ref. [10], and the reader is referred to that work for more detail. Here we use that reference to illustrate the size of various effects in the OPE.

First consider the sizes of the asymptotic elements in the vacuum polarization functions. Referring back to eq. (2.14), we can display the relative size of the coefficients of Q^{-6} and Q^{-8}. We find this to be

$$\frac{\mathcal{E}^{(8)}_{\mu\simeq 2\,\text{GeV}}}{2\pi\langle\alpha_s\mathcal{O}_8^{(6)}\rangle_{\mu\simeq 2\,\text{GeV}}} \simeq -1.5\,\text{GeV}^2\,. \qquad (2.17)$$

Thus the dimension-eight effect is quite relevant for the $\mu \sim 1 \to 2\,\text{GeV}$ region.

Let us also look at the magnitude of the three terms in the OPE shown in eq. (2.16). In the same order as displayed there (and in units of 10^{-7}) we find

$$10^7\,\mathrm{GeV}^{-2}\,\mathcal{M} = \begin{cases} -0.12 - 3.84 + 0.64 + \cdots & (\mu = 1\,\mathrm{GeV}) \\ -0.28 - 3.49 + 0.30 + \cdots & (\mu = 1.5\,\mathrm{GeV}) \\ -0.44 - 3.24 + 0.17 + \cdots & (\mu = 2\,\mathrm{GeV}) \\ -0.89 - 2.63 + 0.04 + \cdots & (\mu = 4\,\mathrm{GeV})\,. \end{cases} \tag{2.18}$$

We see that at $\mu = 1\,\mathrm{GeV}$, the dimension-eight term is larger than the leading operator $\mathcal{O}_1^{(6)}$ in the OPE.[4] Even at $\mu = 2\,\mathrm{GeV}$, it remains a significant size relative to this operator. However, at $\mu = 4\,\mathrm{GeV}$ it is clearly small enough to be neglected.

Through a great deal of effort, the short-distance perturbative structure of the weak interactions has been studied through two-loop order [7, 8]. The dimension-eight effects are large enough that their neglect would negate this effort, as we would be left with only crude evaluations, at least at values of μ which are presently used.

2.3 Separation scheme dependence

If one changes the separation scale, there is mixing between the operators of dimension six and dimension eight. For example, if we use the scale μ_1 instead of the scale μ, the local operator at the new scale is related (in our perturbative treatment) to those at the old one by

$$\langle \mathcal{O}_1^{(6)} \rangle_{\mu_1} = \langle \mathcal{O}_1^{(6)} \rangle_{\mu} + \frac{3}{8\pi} \ln\left(\frac{\mu_1^2}{\mu^2}\right) \langle \alpha_s \mathcal{O}_8^{(6)} \rangle + \frac{3\mathcal{E}_\mu^{(8)}}{16\pi^2} \left(\frac{1}{\mu^2} - \frac{1}{\mu_1^2}\right)\,. \tag{2.19}$$

Thus portions of the dimension-eight effect will appear within the dimension-six operator evaluated at a given scale.

There is also a dependence on the scheme by which one performs the separation of scales. In the above example, we used a sharp cutoff in the variable Q^2 as the method of dividing the low and high energy regions. This is certainly the most convenient method in the context of the present calculation, yet need not be the only possible method. Imagine a smoother cutoff $F(Q^2/\mu^2)$, with $F(Q^2/\mu^2) \to 1$ for $Q^2 \ll \mu^2$ and $F(Q^2/\mu^2) \to 0$ for $Q^2 \gg \mu^2$. We assume that this function is such that all the following integrals are well behaved. Then let us define the vacuum matrix element of $\mathcal{O}_1^{(6)}$ in a so-called 'F-scheme' as

$$\langle \mathcal{O}_1^{(6)} \rangle_\mu^{(\mathrm{F})} \equiv \frac{3}{16\pi^2} \int_0^\infty dQ^2\; Q^4\; F\left(\frac{Q^2}{\mu^2}\right) \left[\Pi_{V,3}(Q^2) - \Pi_{A,3}(Q^2)\right] \tag{2.20}$$

We also define the integrals

$$\begin{aligned} c_F &\equiv -\ln\left(\frac{M_W^2}{\mu^2}\right) + \int_0^\infty \frac{dQ^2}{Q^2}\, \frac{M_W^2}{Q^2 + M_W^2} \left[1 - F\left(\frac{Q^2}{\mu^2}\right)\right] \\ \frac{d_F}{\mu^2} &\equiv \int_0^\infty \frac{dQ^2}{Q^4} \left[1 - F\left(\frac{Q^2}{\mu^2}\right)\right]\,. \end{aligned} \tag{2.21}$$

[4] We will see that the appropriate comparison is with effect of $\mathcal{O}_1^{(6)}$ rather than $\mathcal{O}_8^{(6)}$.

We use these by inserting $1 = F + (1 - F)$ into the full matrix element, eq. (2.10). The first factor gives the $\mathcal{O}_1^{(6)}$ matrix element in the F scheme, and the remaining integrals can be done. In this scheme, the OPE for the amplitude $\mathcal{M}$ reads

$$\mathcal{M} \simeq \frac{G_F}{2\sqrt{2}F_\pi^2}\left[\langle\mathcal{O}_1^{(6)}\rangle_\mu^{(\mathrm{F})} + \frac{3}{8\pi}\left[\ln\left(\frac{M_W^2}{\mu^2}\right) + c_F\right]\langle\alpha_s\mathcal{O}_8^{(6)}\rangle_\mu + \frac{3d_F}{16\pi^2}\frac{\mathcal{E}_\mu^{(8)}}{\mu^2} + \cdots\right]. \tag{2.22}$$

Therefore the matrix elements in the two schemes are related by

$$\langle\mathcal{O}_1^{(6)}\rangle_\mu^{(\mathrm{F})} = \left[\langle\mathcal{O}_1^{(6)}\rangle_\mu^{(\mathrm{c.o.})} - \frac{3c_F}{8\pi}\langle\alpha_s\mathcal{O}_8^{(6)}\rangle_\mu + \frac{3(1-d_F)}{16\pi^2}\frac{\mathcal{E}_\mu^{(8)}}{\mu^2} + \cdots\right]. \tag{2.23}$$

To fully specify the OPE and the matrix elements, one needs to clearly specify the scheme for separating the scales. A lattice evaluation would involve different combinations than does our initial sharp cutoff scheme. We will return to this issue in the next section.

2.4 Dimensional regularization

The presence of dimension-eight operators scaled by an inverse power of μ^2 appears odd in the method of dimensional regularization since one expects only logarithms of μ in that scheme. This is because in dimensional regularization one introduces an energy scale $\mu_{\mathrm{d.r.}}$ ('d.r.' denotes dimensional regularization) in order to maintain the proper dimensions away from $d = 4$. This scale appears only in the form $\mu_{\mathrm{d.r.}}^{d-4}$ and as $d \to 4$, $\mu_{\mathrm{d.r.}}$ will appear only in logarithms. In this section we clarify this issue by means of explicit calculation. It will be seen that there exists a confusion between μ on the one hand and $\mu_{\mathrm{d.r.}}$ on the other. The former is defined to be a separation scale (in the sense of effective field theory or of Wilson's OPE) whereas the latter has nothing to do with the separation of long and short distance. Indeed, dimensional regularization does not itself provide a mechanism for the separation of scales; *all* scales contribute to both the operator and the coefficient functions in such a scheme.

We can evaluate the vacuum polarization functions as defined in d dimensions,

$$\begin{aligned}\mu_{\mathrm{d.r.}}^{d-4} i\int d^dx\ e^{iq\cdot x}\langle 0|T\left(V_3^\mu(x)V_3^\nu(0) - A_3^\mu(x)A_3^\nu(0)\right)|0\rangle &= \\ = (q^\mu q^\nu - q^2 g^{\mu\nu})(\Pi_{V,3} - \Pi_{A,3})(q^2) - q^\mu q^\nu \Pi_{A,3}^{(0)}(q^2)\,.\end{aligned} \tag{2.24}$$

From this follows an expression for the vacuum matrix element of $\mathcal{O}_1^{(6)}$ in dimensional regularization,

$$\begin{aligned}\langle\mathcal{O}_1^{(6)}\rangle_{\mu_{\mathrm{d.r.}}}^{(\mathrm{d.r.})} &\equiv \langle 0|T\left(V_3^\mu(0)V_{\mu,3}(0) - A_3^\mu(0)A_{\mu,3}(0)\right)|0\rangle \\ &= \frac{(d-1)\mu_{\mathrm{d.r.}}^{4-d}}{(4\pi)^{d/2}\Gamma(d/2)}\int_0^\infty dQ^2\ \ Q^d\left(\Pi_{V,3} - \Pi_{A,3}\right)(Q^2)\,.\end{aligned} \tag{2.25}$$

When $d < 4$, this expression is finite. A key point is that the integral continues to run over all Q^2. It is not hard to relate this operator to the one found using a cutoff regularization. To this end, we split the Q^2 integral into regions below and above $Q^2 = \mu^2$. Note that μ is not the same as $\mu_{\rm d.r.}$. For the part of the integration below separation scale μ^2 the integral is finite for all dimensions, and we can take the limit $d \to 4$. This portion of the integration then reproduces exactly the cutoff version of the matrix element. We are left with the difference between the cutoff and dimensional regularization matrix elements. It comes entirely from the high energy region, i.e. with Q above the separation scale μ (see ref. [10] for details; here we use the NDR scheme for γ_5),

$$\begin{aligned}
&\frac{(d-1)\mu_{\rm d.r.}^{4-d}}{(4\pi)^{d/2}\Gamma(d/2)}\int_{\mu^2}^{\infty} dQ^2\; Q^d\,(\Pi_{V,3}-\Pi_{A,3})\,(Q^2) = \\
&\quad = \frac{(d-1)\mu_{\rm d.r.}^{4-d}}{(4\pi)^{d/2}\Gamma(d/2)}\int_{\mu^2}^{\infty} dQ^2\; Q^d \left[\frac{2\pi\langle\alpha_s O_8\rangle}{Q^6}\left(1-\frac{\epsilon}{4}\right)+\frac{\mathcal{E}_\mu^{(8)}}{Q^8}+\ldots\right] \\
&\quad = \frac{3}{8\pi}\langle\alpha_s O_8\rangle\left[\frac{2}{4-d}-\gamma+\ln(4\pi)+\ln\left(\frac{\mu_{\rm d.r.}^2}{\mu^2}\right)-\frac{1}{6}\right]+\frac{3}{16\pi^2}\frac{\mathcal{E}_\mu^{(8)}}{\mu^2}+\cdots. \qquad (2.26)
\end{aligned}$$

In $\overline{\rm MS}$ renormalization, the $2/(4-d) - \gamma + \ln(4\pi)$ terms are removed. Completing the calculation one finds

$$\langle\mathcal{O}_1^{(6)}\rangle_{\mu_{\rm d.r.}}^{(\overline{\rm MS})} = \langle\mathcal{O}_1^{(6)}\rangle_\mu^{(\rm c.o.)} + \frac{3\alpha_s}{8\pi}\left[\ln\left(\frac{\mu_{\rm d.r.}^2}{\mu^2}\right)-\frac{1}{6}\right]\langle\mathcal{O}_8^{(6)}\rangle_\mu + \frac{3}{16\pi^2}\frac{\mathcal{E}_\mu^{(8)}}{\mu^2}\,. \qquad (2.27)$$

Thus, all of the dimension-eight operator is shifted into the $\overline{\rm MS}$ definition of the dimension-six operator. This is seen to be consistent:

1. When one performs a separation of scales, one has the need for dimension-eight operators in the OPE scaled by $1/\mu^2$.

2. When one defines instead the OPE using dimensional regularization, one cannot get effects proportional to $1/\mu_{\rm d.r.}^2$, but the same effect appears contained within the dimension-six operator matrix element. Inspection of eq. (2.16) shows that one obtains the same overall matrix element.[5]

It is important to recognize that dimensional regularization is not a true separation of scales for the OPE in the original sense meant by Wilson. [1, 2, 3, 4] The dimensionally regularized matrix element contains effects from *all* scales, including finite but sizeable contributions from short distances. Thus any use of dimensional regularization for the coefficient functions must be accompanied by an evaluation of the matrix element that covers all scales.

[5] Since we are treating the dimension-six coefficients at leading-log order, we can ignore the nonlogarithic dimension-six portion of eq. (2.27). To include it only requires an inclusion of the nonlogarithmic terms in the coefficient function.

The problem with this situation is that, in present practice, the matrix elements are always calculated with some form of a cutoff but the coefficient functions are calculated dimensionally. This is an inconsistent procedure. To relate operators in a cutoff scheme to dimensional ones, a dimension-eight effect needs to be included, as in eq. (2.27). We will return to a more complete discussion of this point later.

3. Dimension-eight operators in the standard model

In this section we calculate the dimension-eight operators (and their Wilson coefficients) which are relevant for the Standard Model at one loop. We present these first employing a cutoff to provide a separation of scales, then return to a discussion of how to use these results in the context of dimensional regularization.

3.1 Defining a cutoff procedure

The construction of the OPE is performed by comparing a calculation performed both in the full Standard Model and within the effective theory. The coefficients of the effective theory are adjusted such that the results of the two are identical to a given order. For this purpose, calculations using free quarks and gluons are simplest, and are sufficient to identify the operators and their coefficients.

We perform this calculation to one-loop order. For the full Standard Model amplitude, one calculates up to $\mathcal{O}(\alpha_s)$ in the usual way. For the effective theory one takes all matrix elements at tree level, except for those of the leading operator,

$$\mathcal{Q}_2^{(6)} \equiv \bar{u}\Gamma_\mu^L s\, \bar{d}\Gamma_L^\mu u\,, \tag{3.1}$$

which must be calculated at one-loop level in order to properly include the $\mathcal{O}(\alpha_s)$ effects. In both the full and effective theories one adopts the same external states and kinematics. It is important, however, to employ a distinct four-momentum for each of the external states.

We use a cutoff to regularize the matrix element in the following way. Consider the matrix element of a current product at different spacetime points,

$$\mathcal{M}_{A\to B} = \langle B|T(J_\mu^{\rm ch}(x) J_{\rm ch}^{\dagger\mu}(0))|A\rangle\,. \tag{3.2}$$

where $J_{\rm ch}^\mu$ is the hadronic charged weak current. Since the current is a color singlet, $\mathcal{M}_{A\to B}$ is invariant under QCD gauge transformations for any value of x. In order to define an operator matrix element from this, we fourier transform to momentum space, rotate to euclidean momentum and apply a cut-off on the euclidean momentum such that $Q^2 \le \mu^2$,

$$\langle B|O|A\rangle_\mu = \int \frac{d^4Q}{(2\pi)^4}\, \Theta(\mu^2 - Q^2) \int d^4x\; e^{iQx} \langle B|T(J_\mu(x) J^\mu(0))|A\rangle\,. \tag{3.3}$$

This procedure puts the low-Q^2 components into the matrix element and leaves the high-Q^2 portion to be accounted for in the OPE. The analysis of the high-Q^2 portion is then accomplished in the same way as in QCD sum rules via the specification of the momentum flowing in the currents. In practice, this amounts to the following recipe. While calculating insertions in one-loop diagrams, we imagine the two currents to be connected by a W-like boson with virtual four-momentum k. We route the virtual momenta in the loop according to this prescription, choosing k as the loop-integration variable and regularizing the theory by means of a sharp cutoff μ^2 on the euclidean squared-momentum k_E^2. The result obtained this way is UV finite, and we *define* it as the matrix element at the scale μ.

Given this separation of scales, the OPE operators can be calculated in one of two equivalent ways.

1. We first calculate a matrix element in the full theory, which will be finite and independent of μ. The *low energy* radiative corrections to the operator $\mathcal{Q}_2^{(6)}$ below the scale μ are then calculated in the effective theory. While the infrared portions of these matrix elements will be the same, a comparison of the two reveals that specific local operators need to be added to the effective theory in order to reproduce the results of the full theory. These are the new operators in the OPE at one loop. In this method, dimension-eight operators are needed because radiative corrections of the matrix elements in the effective theory contain $1/\mu^2$ effects, which do not occur in the full theory, and hence must be corrected for by dimension-eight operators.

2. The same result can be found in the *high energy* portion of the calculation. In this method, the portion of the full theory that occurs above the scale μ is considered. This portion is equivalent to a set of local operators and the calculation readily reveals their coefficients. [16] In this case, it is seen that dimension-eight operators are a real contribution to the matrix element, and the factor of $1/\mu^2$ arises simply as the lower end of the region of momentum being considered. It is this method which was used in the first portion of this paper.

These two methods are equivalent because the net physics is independent of μ.

3.2 Results

We summarize our calculation of dimension-eight contributions to both the current-current operators and the QCD penguin vertex. *Throughout we work in the chiral limit of* $m_u = m_d = m_s = 0$. This is an appropriate and useful limit because it captures the leading chiral contribution to kaon matrix elements. The leading weak chiral lagrangian that contains factors of the quark masses can be diagonalized away by a chiral rotation. This implies that the effects of quark masses are suppressed by one order in the chiral expansion.

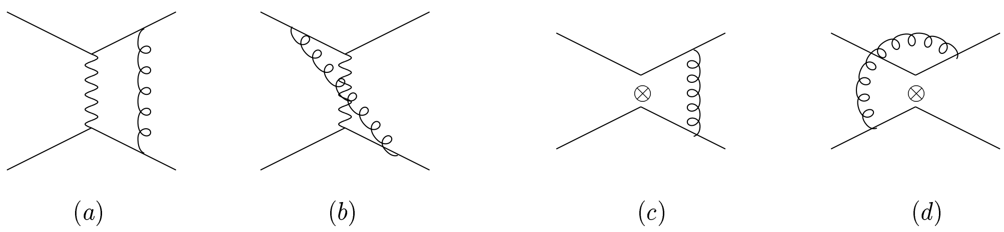


Figure 2: QCD corrections to the box: full theory (a)-(b), effective theory (c)-(d).

Box diagrams: first, we consider the current-current sector of the $\Delta S = 1$ four-quark sector which arises from the box diagram. We depict in figure 2 both the 'full' and 'effective' descriptions for one of the four possible box-diagram contributions. Virtual quarks which appear within a box loop are given a (very small) common mass 'm' to regularize infrared behavior.

To zeroth order in QCD the current-current hamiltonian is expressible entirely in terms of the dimension-six operator $\mathcal{Q}_2^{(6)}$ (cf. eq. (3.1)),

$$\mathcal{H}_{\rm eff}\Big|_{\rm No\ \ QCD} = \frac{G_F}{\sqrt{2}} V_{us} V_{ud}^* \, \mathcal{Q}_2^{(6)} \, . \tag{3.4}$$

The inclusion of QCD to first order yields the familiar dimension-six component, which we express in the 'color-basis' as

$$\mathcal{H}^{(6)}_{\rm (curr-curr)} = \frac{G_F}{\sqrt{2}} V_{us} V_{ud}^* \left[\mathcal{C}_2^{(6)} \mathcal{Q}_2^{(6)} + \mathcal{C}_C^{(6)} \mathcal{Q}_C^{(6)} \right] , \tag{3.5}$$

where

$$\mathcal{Q}_C^{(6)} = \bar{d} \Gamma^\mu_a u \, \bar{u} \Gamma^a_\mu s \tag{3.6}$$

and

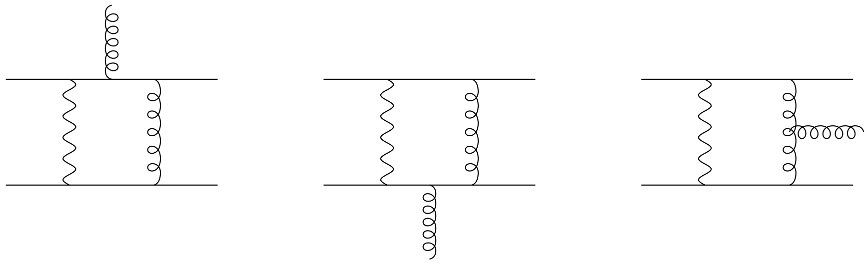
$$\mathcal{C}_2^{(6)}(\mu) = 1 + \mathcal{O}(\alpha_s^2) \, , \qquad \mathcal{C}_C^{(6)}(\mu) = -\frac{3}{2} \frac{\alpha_s}{\pi} \ln\left(\frac{M_W^2}{\mu^2} \right) . \tag{3.7}$$

The set of dimension-eight current-current operators is constructed from four-quark products, covariant derivatives and gluon field-strength tensors. The gluon field-strength tensors we shall employ in our analysis are

$$F^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu - g_3 f_{abc} G^b_\mu G^c_\nu \, , \qquad \tilde{F}^a_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F_a^{\alpha\beta} \, . \tag{3.8}$$

As such, there will be contributions from gluon emission graphs (see figure 3). Also, in constructing the dimension-eight operator basis, we have used the quark equation of motion and current conservation in the chiral limit,

$$\gamma \cdot \mathcal{D} \, q = 0 \, , \qquad \mathcal{D}_\mu \left(\bar{q}_1 \Gamma^\mu q_2 \right) = 0 \, . \tag{3.9}$$



(a) (b) (c)

Figure 3: Gluon radiation from the box diagram.

The list of gauge invariant dimension-eight operators is then[6]

$$
\begin{aligned}
\mathcal{Q}^{(8)}_1 &= \bar u\, \overleftarrow{\mathcal{D}}_\mu \overleftarrow{\mathcal{D}}^\mu\, \Gamma^a_\nu s\, \bar d \Gamma^\nu_a u + \bar u \Gamma^a_\nu \mathcal{D}_\mu \mathcal{D}^\mu s\, \bar d \Gamma^\nu_a u + \bar u \Gamma^a_\nu s\, \bar d\, \overleftarrow{\mathcal{D}}_\mu \overleftarrow{\mathcal{D}}^\mu\, \Gamma^\nu_a u + \bar u \Gamma^a_\nu s\, \bar d \Gamma^\nu_a \mathcal{D}_\mu \mathcal{D}^\mu u \\
\mathcal{Q}^{(8)}_2 &= \bar u \Gamma^a_\nu \mathcal{D}_\mu s\, \bar d \Gamma^\nu_a \mathcal{D}^\mu u + \bar u\, \overleftarrow{\mathcal{D}}^\mu\, \Gamma^a_\nu s\, \bar d\, \overleftarrow{\mathcal{D}}_\mu\, \Gamma^\nu_a u \\
\mathcal{Q}^{(8)}_3 &= \bar u\, \overleftarrow{\mathcal{D}}_\mu\, \Gamma^a_\nu s\, \bar d \Gamma^\nu_a \mathcal{D}^\mu u + \bar u \Gamma^a_\nu \mathcal{D}_\mu s\, \bar d\, \overleftarrow{\mathcal{D}}^\mu\, \Gamma^\nu_a u \\
\mathcal{Q}^{(8)}_4 &= \bar u\, \overleftarrow{\mathcal{D}}_\mu \overleftarrow{\mathcal{D}}_\nu\, \Gamma^\nu_a s\, \bar d \Gamma^\mu_a u + \bar u \Gamma^\nu_a \mathcal{D}_\nu \mathcal{D}_\mu s\, \bar d \Gamma^\mu_a u + \bar u \Gamma^\nu_a s\, \bar d\, \overleftarrow{\mathcal{D}}_\nu \overleftarrow{\mathcal{D}}_\mu\, \Gamma^\mu_a u + \bar u \Gamma^\nu_a s\, \bar d \Gamma^\mu_a \mathcal{D}_\mu \mathcal{D}_\nu u \\
\mathcal{Q}^{(8)}_5 &= \bar u\, \overleftarrow{\mathcal{D}}_\mu\, \Gamma^\nu_a s\, \bar d\, \overleftarrow{\mathcal{D}}_\nu\, \Gamma^\mu_a u + \bar u \Gamma^a_\nu \mathcal{D}_\mu s\, \bar d \Gamma^\mu_a \mathcal{D}^\nu u \\
\mathcal{Q}^{(8)}_6 &= \bar u\, \overleftarrow{\mathcal{D}}_\mu\, \Gamma^\nu_a s\, \bar d \Gamma^\mu_a \mathcal{D}_\nu u + \bar u \Gamma^\nu_a \mathcal{D}_\mu s\, \bar d\, \overleftarrow{\mathcal{D}}_\nu\, \Gamma^\mu_a u \\
\mathcal{Q}^{(8)}_7 &= g_3 \delta^{ab} \tilde F^{\mu\nu,b} \left[\bar u \Gamma^a_\mu s\, \bar d \Gamma_\nu u - \bar u \Gamma_\mu s\, \bar d \Gamma^a_\nu u \right] ,
\end{aligned}
\tag{3.10}
$$

where $\overleftarrow{\mathcal{D}}_\mu$ denotes a left-acting operation and we define the convenient notation

$$\Gamma^a_\mu \equiv \frac{\lambda^a}{2} \Gamma^L_\mu \equiv \frac{\lambda^a}{2} \gamma_\mu (1 + \gamma_5)\, . \tag{3.11}$$

We find the Wilson coefficients corresponding to the above dimension-eight operators to have the form

$$\mathcal{C}^{(8)}_i = \frac{\alpha_s}{4\pi} \cdot \frac{1}{\mu^2} \cdot \eta^{(8)}_i \, , \tag{3.12}$$

where the $\{\eta^{(8)}_i\}$ coefficients are:

$\eta^{(8)}_1$	$\eta^{(8)}_2$	$\eta^{(8)}_3$	$\eta^{(8)}_4$	$\eta^{(8)}_5$	$\eta^{(8)}_6$	$\eta^{(8)}_7$
5/3	22/3	8/3	−1/3	16/3	14/3	1/3

(3.13)

QCD penguin vertex: in the chiral limit, the general form for the QCD penguin effective vertex is

$$\mathcal{H}^{(\mathrm{QCD-pgn})}_{\mathrm{eff}} = \frac{G_F}{\sqrt{2}} \cdot V_{us} V^*_{ud} \left[\mathcal{C}^{(6)}_{\mathrm{P}} \mathcal{O}^{(6)}_{\mathrm{P}} + \sum_{i=1}^{3} \mathcal{C}^{(8)}_{\mathrm{P}i} \mathcal{O}^{(8)}_{\mathrm{P}i} + \cdots \right] . \tag{3.14}$$

[6]It might appear that the list in eq. (3.10) is missing another possible dimension-eight operator, namely $g_3 f_{abc} F^{\mu\nu,b} \bar u \Gamma^a_\nu s\ \bar d \Gamma^c_\mu u$. However, one can show that in the chiral limt, this is expressible in terms of $\mathcal{Q}^{(8)}_5$ and $\mathcal{Q}^{(8)}_6$.

Figure 4: QCD penguin vertex in the full theory.

Here, we employ a two-generation approximation (no dependence on the virtual top quark in the penguin loop). The penguin vertex as it appears in the 'full' theory is displayed in figure 4. Observe the presence of self-energy graphs in figures 4*b* and 4*c*.

The $d = 6$ QCD-penguin operator and its Wilson coefficient are given (in the chiral limit) by

$$\mathcal{O}_{\rm P}^{(6)} = \bar{d}\Gamma_{\nu}^{a} s\, \mathcal{D}_{\mu} F_{a}^{\mu\nu} , \qquad \mathcal{C}_{\rm P}^{(6)} = \frac{g_3}{(4\pi)^2}\left[-\frac{4}{3}\ln\frac{\mu^2+m_c^2}{\mu^2} + 2g_c - 2g_c^2 + \frac{2}{9}g_c^3\right], \tag{3.15}$$

where g_3 is the QCD coupling constant and g_c is the dimensionless quantity

$$g_c \equiv \frac{m_c^2}{\mu^2+m_c^2} . \tag{3.16}$$

Proceding next to the dimension-eight component, there are two classes of local, gauge-invariant operators: with an $s \to d$ quark bilinear and one field-strength tensor ($\mathcal{O}_{\rm P1}^{(8)}$) and with an $s \to d$ quark bilinear and two field-strength tensors ($\{\mathcal{O}_{{\rm P}i}^{(8)}\}$ ($i = 2, 3$)). The latter correspond to two-gluon emission as in figure 4*d*. Thus we find

$$\begin{aligned} \mathcal{O}_{\rm P1}^{(8)} &= \bar{d}\Gamma_{\nu}^{a} s\, \mathcal{D}^{\alpha}\mathcal{D}_{\alpha}\mathcal{D}_{\mu} F_{a}^{\mu\nu} \\ \mathcal{O}_{\rm P2}^{(8)} &= i\bar{d}\left[\frac{\lambda^a}{2}, \frac{\lambda^b}{2}\right]\Gamma_{L}^{\mu} s\, F_{b}^{\alpha\beta}\mathcal{D}_{\mu}F_{\alpha\beta}^{a} \\ \mathcal{O}_{\rm P3}^{(8)} &= \bar{d}\left[\frac{\lambda^a}{2}, \frac{\lambda^b}{2}\right]_{+}\Gamma_{L}^{\mu} s\, F_{a}^{\alpha\beta}\mathcal{D}_{\alpha}\tilde{F}_{\beta\mu}^{b} , \end{aligned} \tag{3.17}$$

along with the $d = 8$ Wilson coefficients,

$$\begin{aligned} \mathcal{C}_{\rm P1}^{(8)}(\mu) &= \frac{g_3}{(4\pi)^2}\frac{1}{\mu^2+m_c^2}\left[\frac{1}{3}\frac{m_c^2}{\mu^2} - \frac{2}{3}g_c + \frac{4}{3}g_c^2 - \frac{2}{3}g_c^3 + \frac{1}{15}g_c^4\right] \\ \mathcal{C}_{\rm P2}^{(8)}(\mu) &= \frac{\alpha_s}{3\pi}\frac{1}{\mu^2+m_c^2}\left[-\frac{4}{3}\frac{m_c^2}{\mu^2} - 3g_c + \frac{10}{3}g_c^2 - \frac{13}{6}g_c^3 + \frac{2}{5}g_c^4\right] \\ \mathcal{C}_{\rm P3}^{(8)}(\mu) &= \frac{\alpha_s}{3\pi}\frac{1}{\mu^2+m_c^2}\left[\frac{4}{3}\frac{m_c^2}{\mu^2} + 2g_c - \frac{4}{3}g_c^2 - \frac{1}{3}g_c^3\right] . \end{aligned} \tag{3.18}$$

3.3 Conversion to dimensional regularization

As we have stated, the dimension-eight operators and Wilson coefficients of the previous section are defined in terms of a particular momentum cutoff procedure. It is essential to understand how these relate to those obtained via other possible calculational approaches, most notably dimensional regularization and $\overline{\rm MS}$ renormalization.

In dimensional regularization, there are no $1/\mu^2$ effects. Therefore all dimension-eight effects must be transfered from existing as explicit operators in the OPE to being contained within some $\overline{\mathrm{MS}}$ matrix element. This was previously illustrated in our sample calculation section 2 as embodied in eq. (2.27). In the present case we are looking at radiative corrections to the operator $\mathcal{Q}_2^{(6)}$ and therefore at this order all the dimension-eight effects will be absorbed into the $\overline{\mathrm{MS}}$ matrix element of $\mathcal{Q}_2^{(6)}$. It has long been realized that to convert to $\overline{\mathrm{MS}}$ from some form of cutoff in the evaluation of matrix elements, a mixing of dimension-six operators is needed. What is new in the present work is the realization that dimension-eight physics is also needed. In terms of the coefficients $\{\mathcal{C}_i^{(8)}\}$ calculated above, the relation is

$$\langle \mathcal{Q}_2^{(6)} \rangle_\mu^{\overline{\mathrm{MS}}} = \langle \mathcal{Q}_2^{(6)} \rangle_\mu^{c.o} + \sum_i d_i \langle \mathcal{Q}_i^{(6)} \rangle_\mu + \sum_i \mathcal{C}_i^{(8)} \langle O_i^{(8)} \rangle \, . \tag{3.19}$$

Here d_i are mixing coefficients for dimension six which we do not calculate in this paper. However, the key new feature here is that the matrix elements of the dimension-eight operators must be added directly to that of dimension six in order to form the $\overline{\mathrm{MS}}$ matrix element. Other $\overline{\mathrm{MS}}$ operators will also have similar relations involving dimension-eight effects when converting from cutoff schemes. We discuss this point in the appendix.

4. Conclusion

We have uncovered a basic problem with existing calculations of weak amplitudes. The coefficients in the OPE are calculated using dimensional regularization and hence do not include dimension-eight operators. However, all matrix elements are calculated with some variation of a cutoff and hence also do not contain the effects of dimension-eight operators. When one connects the matrix elements to those in the $\overline{\mathrm{MS}}$ scheme, one needs to consider *both* dimension-six and dimension-eight operators as our example shows in eq. (2.27). However in practice this is not done, and the result is then inconsistent.

Lattice evaluations of operators [17] rely on the finite lattice spacing to remove all short distance physics. This is a true cutoff, as the effects of short distance physics is simply not present in the simulation. Most current lattice calculations utilize the inverse lattice spacing as the scale μ that defines the matrix element, although there are alternative possibilities (see item 1 below for an example). Therefore, to convert from lattice regularization at any fixed lattice spacing to a dimensional scheme requires the addition of both dimension-six and dimension-eight operators. This can be done as part of the 'improvement' procedure [20] which attempts to correct for the effects of lattice artifacts, including the effects of finite lattice spacing. On the lattice the effects of higher dimension operators can be more severe, as the

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lack of chiral invariance can generate a dimension seven operator,[7] whose effect also needs to be corrected for. The present state of the art does not yet include the effects of dimension eight operators, but in the future it should be possible to correct also for these.

Quark model (and also large N_c) matrix elements are in a far more difficult situation [18]. In these cases, low energy models are postulated, and the models only make sense below 1 GeV or so. Many of these models are treated with a cutoff of order 1 GeV, and thus at best contain an incomplete description of the required short-distance physics. While lattice methods are correctible to account for dimension-eight effects, the same appears doubtful in quark model methods. We then must conclude that there are very large intrinsic uncertainties associated with these methods.

There are several strategies for overcoming the basic inconsistency we have revealed:

1. In principle, the best way is avoid the need for a cutoff in the matrix element calculation — to have a method which is sensitive to all scales. This can then directly yield $\overline{\mathrm{MS}}$ matrix elements. One possible procedure is the lattice analysis of the kaon B-parameter in ref. [19], where the scale $\mu_{\mathrm{d.r.}}$ is held fixed while the lattice spacing is varied. The various values of the lattice spacing are then used to perform an extrapolation to the continuoum limit, at fixed $\mu_{\mathrm{d.r.}}$. Although this was done to remove lattice artifacts, it also has the effect of removing the need for dimension-eight effects (since the separation scale in the matrix element is sent to infinity). A second procedure is the evaluation that we provided in the first half of this paper, based on sum rule techniques where one knows the vacuum polarization functions at all scales [10, 11]. In this method there is no obstacle to a full evaluation, because we can take the separation scale to infinity.[8] There exist still other possible procedures. In ref. [21], it has been suggested (although not yet implemented) that the weak currents comprising the weak hamiltonian be evaluated at separation scale x small enough that perturbation theory be valid but large enough that lattice artifacts be suppressed. A second lattice-theoretic approach would involve decoupling the lattice spacing a from the scale μ by using the inverse lattice volume to set the scale. [22] Finally, the work of ref. [23] describes a non-lattice calculation which is sensitive to all distance scales.

   In general, we feel that the lattice seems most suited for a systematic program of extrapolating matrix elements to very large separation scales. We caution, however, that this has not yet been done in all cases of interest.

[7] We thank Chris Sachrajda for pointing out the relevance of the lattice imporvement program and for discussions on these points.

[8] This will be discussed more fully in a separate work by us.

2. A less ambitious but still valuable strategy would be to work at values of separation scale μ large enough that the problems of higher dimensions are numerically insignificant. Stated in a different way, at large enough μ the matrix element evaluation will already contain all the needed ingredients, and residual dimension-eight effects that are missing will be small enough. This option is likely only available for lattice regularization, where it is tied to the ability to decrease the lattice spacing. Quark model methods make sense only at low energy and cannot be extended to larger μ. The question in this case is how large μ must be for a result of a given accuracy. We have addressed this for our model hamiltonian above with the result that $\mu \sim 3 - 4\,\mathrm{GeV}$ appears to be required. If the lattice is used to give estimates of matrix element of the dimension eight operators calculated in section 3, these results could be used to provide a second estimate of the relative importance of these terms. An advantage of this method is that one does not need to be highly accurate or highly consistent. If the dimension-eight effects are small at a given μ, then we do not need to match them to the rest of the calculation. We instead use the estimated size to produce an error bar due to this effect. If the error bar is small enough, we need need not consider dimension-eight operators further.

3. Alternatively, one might adopt a cut-off scheme for both the matrix elements and the OPE coefficients. This would bring dimension-eight operators into the OPE, and their matrix elements would also need to be computed. However, this option appears difficult to carry out to the accuracy that we desire. At one loop, a cut-off scheme is only moderately difficult. One needs to be careful to preserve gauge invariance and other symmetres. One also needs to find a scheme that is equally useful for the perturbative calculation of the coefficients and for the calculation of the matrix elements if one is going to match them consistently. This latter requirement is likely difficult. However, it will be extremely difficult to implement a cut-off scheme at two loops. The presence of a dimensionful parameter upsets the normal power counting, and requires great care in properly defining the operators beyond one loop. In addition, the separation of nested loops in a cutoff scheme is subtle and can potentially lead to troubles with gauge invariance and other symmetries.

4. Yet another option would be to continue to follow the most standard practice (of using dimensional regularization for the Wilson coefficients and a cutoff procedure for the operator matrix elements), but to correct the matrix elements to include the short distance effects from dimension-eight operators. In this case one would calculate both dimension-six and dimension-eight operators in some cutoff scheme, then add the contributions together with the right coefficient to form the dimensionally regularized matrix element. This is what is done earlier

in eq. (2.27). This allows one to use the extensive work that has been performed calculating the OPE coefficients. It remians to be seen if this procedure can be successfully carried out in all cases of interest.

Much of the existing work in the field is done at low values of μ. Various quark model and large-N_c methods use $\mu \sim 0.5 \to 1\,\mathrm{GeV}$, at which scales these effects are apparently extremely important. These calculations must be considered to contain enormous uncertanties, at least until further work is done. Lattice calculations are typically carried out with $\mu = 2\,\mathrm{GeV}$, although there is recent progress at working at higher μ. At this scale, dimension-eight contributions appear to still be larger than other effects which are included, such as scheme dependence and two-loop evaluation of the coefficient functions. We look forward to future work that allows us to eliminate or reduce the undertainties that come from the presence of dimension-eight effects.[9]

Acknowledgments

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A. On the matching of cutoff and dimensional regularization procedures

In section 3, we utilized the dimension-six local operator $\mathcal{Q}_2^{(6)}$ in our discussion of higher dimension effects in the Standard Model. Of course, a similar analysis can be performed for other operators $\mathcal{Q}_i^{(6)}$ appearing in the weak hamiltonian. Here, we consider briefly the issue of perturbative matching between cutoff and dimensional regularization approaches in a general situation.

Let us first establish the notation. We let $\mathcal{Q}_i^{(6)}(\mu)$ represent the $d = 6$ operators normalized in any cutoff scheme at the scale μ and $\mathcal{Q}_i^{(6)\overline{\mathrm{MS}}}(\mu_{\overline{\mathrm{MS}}})$ represent the corresponding dimension-six operator in the $\overline{\mathrm{MS}}$ normalization. Further, we let $S_i^{(d)}$

[9]After this work was submitted to the archive, Jan Eeg kindly informed us of some related work. In ref. [24] Penin and Pivovarov discuss dimension-eight operators which appear at order $1/m_c^2$. These are distinct from the $1/\mu^2$ effects that we have discussed. If one is working at values of μ below the mass of the charm quark, the operators discussed by Penin and Pivovarov should be taken into account even in the $\overline{\mathrm{MS}}$ renormalization scheme. Also, in ref. [25] Melsom discusses $1/\mu^2$ dimension-eight operators within a cut-off model. He uses a cutoff procedure which differs from ours (and thus precludes direct comparison with our work), improperly mixes dimensional regularization and a cutoff, and has missed the operators with extra gluons. However, he does correctly point out that dimension-eight operators can have a physical effect at low values of μ.

denote a tree-level matrix element ($S_i^{(d)} \equiv \langle \mathcal{Q}_i^{(d)} \rangle^{(\text{tree})}$) and consider matrix elements taken between states with generic momentum p and quark mass m_q.

Then, calculating in a cutoff scheme, and keeping terms up to dimension eight will give generally

$$\langle \mathcal{Q}_i^{(6)} \rangle_\mu^{(\text{c.o.})} = \left[\delta_{ij} + \alpha_s \gamma_{ij} \ln \left(\frac{\mu^2}{p^2} \right) + \alpha_s f_{ij}^{\text{c.o.}} \right] S_j^{(6)} + \alpha_s \left[c_{ij} + \frac{1}{\mu^2} \tilde{\gamma}_{ij} \right] S_j^{(8)} \,. \qquad \text{(A.1)}$$

Here γ_{ij} is the usual dimension-six anomalous dimension matrix. The coefficients c_{ij} scale as p^{-2} or m_q^{-2} and therefore are sensitive to the infrared (IR) behavior of the matrix element, while $\tilde{\gamma}_{ij}$ is associated with the behavior at the upper end of the integration domain. $f_{ij}^{\text{c.o.}}$ are finite terms depending on the specific scheme adopted for separating the scales. For finite μ the matrix element $\langle \mathcal{Q}_i^{(6)} \rangle_\mu^{(\text{c.o.})}$ will itself be finite and could be used as the definition of the operator matrix element at scale μ. Of course, the operators defined in this way differ from the $\overline{\text{MS}}$ ones by a finite normalization.

We now present the connection to the operators in the $\overline{\text{MS}}$ scheme. This requires an appropriate matching in which dimension-eight operators will appear. An example of this is provided by eq. (2.27) in the study of the LR hamiltonian. Consider then a matrix element of the $d = 6$ operator $\mathcal{Q}_i^{(6)}$ taken between four-quark states. Expressed schematically, the calculation as performed in dimensional regularization ('d.r.') gives

$$\langle \mathcal{Q}_i^{(6)} \rangle_{\mu_{\text{d.r.}}}^{(\text{d.r.})} = \left[\delta_{ij} + \alpha_s \gamma_{ij} \left(\ln \frac{\mu_{\text{d.r.}}^2}{p^2} + \frac{1}{\hat{\epsilon}} \right) + \alpha_s f_{ij}^{\text{d.r.}} \right] S_j^{(6)} + \alpha_s c_{ij} S_j^{(8)} \,, \qquad \text{(A.2)}$$

with

$$\frac{1}{\hat{\epsilon}} \equiv \frac{2}{\epsilon} - \gamma + \ln(4\pi) \,. \qquad \text{(A.3)}$$

The infrared coefficients c_{ij} are the same as in the case of the cutoff scheme. In this case, however, the terms proportional to $\tilde{\gamma}_{ij}$ disappear because in dimensional regularization the integration runs over all scales. Introducing an $\overline{\text{MS}}$-subtracted operator,

$$\langle \mathcal{Q}_i^{(6)} \rangle^{\overline{(\text{MS})}} \equiv \left[\delta_{ij} - \alpha_s \gamma_{ij} \frac{1}{\hat{\epsilon}} \right] \langle \mathcal{Q}_j^{(6)} \rangle^{(\text{d.r.})} \,, \qquad \text{(A.4)}$$

we have

$$\langle \mathcal{Q}_i^{(6)} \rangle_{\mu_{\overline{\text{MS}}}}^{\overline{(\text{MS})}} = \left[\delta_{ij} + \alpha_s \gamma_{ij} \ln \left(\frac{\mu_{\overline{\text{MS}}}^2}{p^2} \right) + \alpha_s f_{ij}^{\text{d.r.}} \right] S_j^{(6)} + \alpha_s c_{ij} S_j^{(8)} \,. \qquad \text{(A.5)}$$

The general form of the above mentioned connection at order α_s is given by

$$\langle \mathcal{Q}_i^{(6)} \rangle_{\mu_{\overline{\text{MS}}}}^{\overline{(\text{MS})}} = \left[\delta_{ij} - \alpha_s \gamma_{ij} \ln \left(\frac{\mu^2}{\mu_{\overline{\text{MS}}}^2} \right) + \alpha_s \left(f_{ij}^{\overline{\text{MS}}} - f_{ij}^{\text{c.o.}} \right) \right] \langle \mathcal{Q}_j^{(6)} \rangle_\mu^{(\text{c.o.})} - \alpha_s \frac{\tilde{\gamma}_{ij}}{\mu^2} \langle \mathcal{O}_j^{(8)} \rangle_\mu \,. \qquad \text{(A.6)}$$

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The right hand side of eq. (A.6) is constructed in such a way to be finite in the limit in which $\mu \to \infty$. The f_{ij} matrices contain finite parts which depend upon the particular scheme adopted both in dimensional regularization and in the cutoff regularization. In particular, $f_{ij}^{\overline{\rm MS}}$ depends on the scheme definition adopted for γ_5 away from dimension four. The coefficients $\tilde{\gamma}_{ij}$ govern the 'leakage' of dimension-eight operators into the dimension-six sector. They depend on the particular scheme adopted for separating scales and, as shown before, they can be obtained by a one-loop calculation in the cutoff scheme.

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